

# Spectral geometry of $\kappa$ -Minkowski space

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After recalling Snyder's idea [1] of using vector fields over a smooth manifold as 'coordinates on a noncommutative space', we discuss a two dimensional toy-model whose 'dual' noncommutative coordinates form a Lie algebra: this is the well known  $\kappa$ -Minkowski space [2].

We show how to improve Snyder's idea using the tools of quantum groups and noncommutative geometry.

We find a natural representation of the coordinate algebra of  $\kappa$ -Minkowski as linear operators on an Hilbert space (a major problem in the construction of a physical theory), study its 'spectral properties' and discuss how to obtain a Dirac operator for this space.

We describe two Dirac operators.

The first is associated with a spectral triple. We prove that the cyclic integral of M. Dimitrijevic et al. [5] can be obtained as Dixmier trace associated to this triple.

The second Dirac operator is equivariant for the action of the quantum Euclidean group, but it has unbounded commutators with the algebra.

## I. INTRODUCTION

In classical mechanics, Legendre transformation in the Hamiltonian formalism exhibits an evident symmetry between coordinate space and momentum space.

The same holds in quantum mechanics on a flat space ( $\mathbb{R}^n$  or a quotient by a discrete subgroup), with Fourier transform/series replacing the Legendre transformation.

Dealing with quantum mechanics on a curved space  $M$  (a smooth manifold), the situation changes. While coordinate functions on  $M$  are elements of a commutative algebra, ‘momenta’ (we mean, vector fields over  $M$ ) no longer commute. A basic example is the 3-sphere, whose vector fields generate  $su(2)$ , the algebra of angular momenta of quantum mechanics.

Snyder’s idea was to use ‘momenta’, i.e. vector fields over a smooth manifold  $M$ , as “coordinates on a noncommutative space” dual to  $M$ .

He applied this idea to the four-dimensional deSitter space, and was led to study the algebra generated by ten operators  $\{x_\mu, J_{\mu\nu}\}$  satisfying the commutation rules (Greek letters run over 0, 1, 2, 3):

$$[x_\mu, x_\nu] = i\lambda^2 J_{\mu\nu} \quad , \quad (1)$$

where  $J_{\mu\nu}$  is an element of  $so(3, 1)$  with suitable commutators with the “coordinates”  $x_\mu$ . One can notice that this algebra (isomorphic to  $so(3, 2)$ ) has too much generators to represent a four-dimensional ‘noncommutative space’, while the elements  $x_\mu$  alone are correct in number, but don’t close an algebra.

A characteristic feature is the presence in (1) of an invariant length  $\lambda$ , whose role is to provide the correct physical dimensions and to allow us to recover  $\mathbb{R}^4$  as  $\lambda$  goes to zero (‘long distances’ or ‘low energy’ limit).

The purpose of this notes is to show how to modify Snyder’s idea in such a way to obtain an algebra that can be studied with noncommutative-geometric tools.

In sec. II, we discuss Snyder’s model in full detail in the two dimensional version, and realize that Snyder’s coordinates don’t close an algebra, neither in two dimensions, nor in four.

In sec. III, we retry in three dimensions, and realize a fundamental difference with the original Snyder example: in the three dimensional case the ‘dual’ of the deSitter space is an algebra. This is a basic property, if we want to study noncommutative geometry [6, 7, 8].

The geometrical reason at the base of the difference between Snyder’s model in three and in  $n \neq 3$  dimensions is clear: the deSitter space in three dimensions is a Lie group, hence the tangent bundle trivialize and vector fields form a Lie algebra isomorphic to the Lie algebra of the group.

In sec. IV, we search for a two-dimensional example and realize that there is only one, the spacetime studied in literature under the name of  $\kappa$ -Minkowski [2], whose coordinate algebra is isomorphic to  $U(sb(2, \mathbb{R}))$ .

As a byproduct of this section, we find a natural representation of the algebra as linear operators on a Hilbert space (the choice of a representation on a Hilbert space is a major problem in the construction of a physical theory). Another main point of this section is the realization of Poincaré as a symmetry of  $\kappa$ -Minkowski, in the spirit of Snyder’s idea.

In sec. V we adopt a different point of view and, applying Weyl quantization, construct a  $C^*$ -algebra encoding informations on the ‘topology’ of the underlying  $\kappa$ -Minkowski space.

We then focus the attention on the construction of a spectral triple over  $\kappa$ -Minkowski space. A proposal for a Dirac operator on  $\kappa$ -Minkowski already appeared in [3], although the Dirac operator they found doesn’t satisfy Connes axioms for a spectral triple (the same

holds for the one parametric family of Dirac operators in [4]). For the generalization of such axioms to the non-compact case we refer to the original paper of A. Connes [7] (see also [9, 10] for a comprehensive presentation).

In section VC, we construct a Dirac operator that fulfil all axioms of a spectral triple, and prove that the Dixmier trace associated to the spectral triple is just the cyclic integral discussed in [5].

Then, we compute a Dirac operator imposing equivariance for the action of the quantum Euclidean group. This turns out to be very similar to the one in [3], and has not bounded commutators with the algebra.

## II. THE TWO-DIMENSIONAL ANALOGUE OF SNYDER'S SPACETIME

In his original article, Snyder considers the deSitter space  $SO(3, 2)/SO(3, 1)$  and identifies the spacetime “coordinate functions” with a basis of the vector subspace of  $so(3, 2)$  orthogonal to  $so(3, 1)$ . The (connected component of the)  $SO(3, 1)$  subgroup provides the Lorentz symmetries and, with a suitable choice of the momenta, it can be extended to a Poincaré symmetry. For the sake of simplicity, we will study, in full details, the two-dimensional version. The interested readers can find the four-dimensional model discussed in Snyder's original article [1].

In the two-dimensional version, the deSitter space is  $SO(2, 1)/SO(1, 1)$ . If  $\eta_0, \eta_1, \eta_2$  are coordinates of  $\mathbb{R}^3$ , the deSitter spacetime is the  $SO(2, 1)$ -orbit of equation

$$\eta_0^2 - \eta_1^2 - \eta_2^2 = -1 \quad . \quad (2)$$

The (three dimensional Lorentz) Lie algebra  $so(2, 1)$  has generators  $\{J, K_1, K_2\}$  (a rotation and two boosts) given by

$$\begin{aligned} J &= i(\eta_2 \partial_1 - \eta_1 \partial_2) \quad , \\ K_1 &= i(\eta_0 \partial_1 + \eta_1 \partial_0) \quad , \\ K_2 &= i(\eta_0 \partial_2 + \eta_2 \partial_0) \quad , \end{aligned}$$

where  $\partial_\mu = \partial/\partial\eta_\mu$ . So, the commutation rules are:

$$[J, K_1] = iK_2 \quad , \quad [J, K_2] = -iK_1 \quad , \quad [K_1, K_2] = -iJ \quad . \quad (3)$$

Emulating Snyder, we call  $N = -K_1$  the generator of the  $so(1, 1)$  subalgebra (the one-dimensional Lorentz Lie-algebra, isomorphic to  $\mathbb{R}$ ) that leave fixed the point  $m_0 = (0, 0, 1)$ , and  $x = \lambda J$ ,  $t = \lambda K_1$  the remaining generators (whose span is the subspace of vector fields over  $SO(2, 1)$  that are tangent to  $M$  at  $m_0$ ). The length  $\lambda$  provides the correct physical dimensions. By (3), the coordinates satisfy the commutation rule

$$[x, t] = i\lambda^2 N \quad ,$$

and clearly don't form a subalgebra of  $so(2, 1)$ .

The higher-dimensional analogue is the commutator  $[x_\mu, x_\nu] = i\lambda^2 J_{\mu\nu}$  anticipated in the introduction.

The action of the Lorentz algebra  $so(1, 1)$  on the coordinates is via commutator, and by (3) is undeformed:

$$N \triangleright x = it \quad , \quad N \triangleright t = ix \quad .$$

Since the action is undeformed, the invariant is the classical quadratic element  $x^2 - t^2$ .

We need to introduce translations, yet.

Let us recall the idea of Snyder: to start with a commutative spacetime, in which translations do not commute, and define a noncommutative spacetime with commutative momenta. Half of this idea was already applied, to find the noncommutative spacetime “dual” to deSitter.

To complete the picture, following Snyder, we define “momenta” as  $P = \lambda^{-1}\eta_1$  and  $E = \lambda^{-1}\eta_0$  (the presence of  $\lambda$  ensures the correct physical dimensions). Together with  $N$  they form the classical Poincaré algebra:

$$[E, P] = 0 \quad , \quad [N, P] = -iE \quad , \quad [N, E] = -iP \quad .$$

The “momenta” act on the coordinates via commutators. For example:

$$P \triangleright x = [P, x] = -i\lambda(\eta_2\partial_1 - \eta_1\partial_2)\lambda^{-1}\eta_1 = -i\eta_2 = -i\sqrt{\lambda^2(E^2 - P^2) - 1} \quad .$$

In the last step we used the equation (2), defining the deSitter space.

With a straightforward calculation we derive the (deformed) action of momenta on coordinates

$$P \triangleright x = -i\sqrt{\lambda^2(E^2 - P^2) - 1} = E \triangleright t \quad , \quad P \triangleright t = E \triangleright x = 0 \quad ,$$

while the associated phase-space is defined by

$$[x, P] = i\sqrt{\lambda^2(E^2 - P^2) - 1} = [t, E] \quad , \quad [t, P] = [x, E] = 0 \quad .$$

In the  $\lambda \rightarrow 0$  limit, the spacetime reduces to a commutative one, phase-space becomes the Heisenberg algebra and the action of the Poincaré algebra reduces to the standard one; so for  $\lambda = 0$  we recover the classical scenario.

### III. THE THREE-DIMENSIONAL ANALOGUE OF SNYDER’S SPACETIME

In two dimensions, we have just seen that “Snyder’s coordinates” don’t form a complete set of generators for an algebra. Let us try in three dimensions.

The space we consider here is the  $SO(2, 2)$ -orbit  $M \subset \mathbb{R}^4$  of equation:

$$\eta_0^2 + \eta_1^2 - \eta_2^2 - \eta_3^2 = -1 \quad , \quad \eta_\mu \in \mathbb{R}^4 \quad . \quad (4)$$

The stability group of the point  $m_0 = (0, 0, 0, 1)$  is  $SO(2, 1)$ , so the orbit is the quotient  $M \simeq SO(2, 2)/SO(2, 1)$ . Being an homogeneous  $SO(2, 2)$ -space, elements of the Lie algebra  $so(2, 2)$  (i.e., vector fields over  $SO(2, 2)$ ) correspond to derivatives of  $C^\infty(M)$ . Vectors of the subspace  $so(2, 1) \subset so(2, 2)$  are orthogonal to  $M$ , so the “naive” approach would be to take a basis of  $so(2, 1)^\perp$ , the orthogonal of  $so(2, 1)$  in  $so(2, 2)$ , as coordinates of the noncommutative spacetime dual to  $M$ . But  $so(2, 1)^\perp$  is not a subalgebra of  $so(2, 2)$ , this because  $M$  is not a group with the quotient structure ( $SO(2, 1)$  is not a *normal* subgroup of  $SO(2, 2)$ ).

Despite this, we can put a Lie group structure on  $M$ . Writing

$$g = \begin{pmatrix} \eta_0 + \eta_3 & \eta_1 + \eta_2 \\ \eta_1 - \eta_2 & -\eta_0 + \eta_3 \end{pmatrix} \quad , \quad \eta_\mu \in \mathbb{R}^4 \quad , \quad (5)$$

we have the obvious (smooth manifold) isomorphism  $M \simeq SL(2, \mathbb{R})$ , the equation  $\det g = 1$  (that identifies  $SL(2, \mathbb{R})$  inside  $\text{Mat}(2, \mathbb{R})$ ) being equivalent to (4). Thus,  $M$  is a double cover of the Lorentz group  $SO(2, 1)$ .

This is the pseudo-Euclidean analogue of the fibration  $SO(4) \xrightarrow{SO(3)} S^3 \simeq SU(2)$ , i.e.  $SO(2, 2) \xrightarrow{SO(2, 1)} SL(2, \mathbb{R})$ .

Since  $M$  is a Lie group, its tangent bundle is trivial. The Lie algebra of global vector fields on  $M$ , isomorphic to  $sl(2, \mathbb{R})$ , will be identified with the algebra “coordinate functions on the noncommutative spacetime”, three dimensional analogue of the Snyder’s spacetime. Let us compute it explicitly.

We take  $\tilde{L} \in sl(2, \mathbb{R})$  (real traceless matrices) and call  $L$  the associated vector field on  $M$ , defined by:

$$(Lf)(g) = \frac{d}{d\tau} \Big|_{\tau=0} f(\exp\{\tau L\} \cdot g) , \quad (6)$$

for all  $f \in C^\infty(M)$ ,  $g \in SL(2, \mathbb{R})$ . We fix a basis for  $sl(2, \mathbb{R})$ :

$$\tilde{t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \tilde{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tilde{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

compute the exponentials

$$\begin{aligned} \exp\{\tau \tilde{t}\} &= \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix} , \\ \exp\{\tau \tilde{x}\} &= \begin{pmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{pmatrix} , \\ \exp\{\tau \tilde{y}\} &= \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} , \end{aligned}$$

and their action via left multiplication on (5), the generic element of  $SL(2, \mathbb{R})$ . Then, through (6), we determine the associated vector fields. For example:

$$\exp\{\tau \tilde{t}\}g = \begin{pmatrix} (\eta_0 + \eta_3)e^\tau & (\eta_1 + \eta_2)e^\tau \\ (\eta_1 - \eta_2)e^{-\tau} & (-\eta_0 + \eta_3)e^{-\tau} \end{pmatrix} .$$

So  $\exp\{\tau \tilde{t}\}$  maps  $\eta$  to the point  $\eta^{(\tau)}$ , with coordinates

$$\begin{pmatrix} \eta_0^{(\tau)} \\ \eta_1^{(\tau)} \\ \eta_2^{(\tau)} \\ \eta_3^{(\tau)} \end{pmatrix} = \begin{pmatrix} \eta_0 \cosh \tau + \eta_3 \sinh \tau \\ \eta_1 \cosh \tau + \eta_2 \sinh \tau \\ \eta_1 \sinh \tau + \eta_2 \cosh \tau \\ \eta_0 \sinh \tau + \eta_3 \cosh \tau \end{pmatrix} ,$$

and by (6):

$$(-i\lambda^{-1}t \cdot f)(\eta) = \frac{d}{d\tau} \Big|_{\tau=0} f(\eta^{(\tau)}) ,$$

where  $\lambda$  is a parameter with the dimensions of a length.

We have called  $-i\lambda^{-1}t$  the derivation associated to  $\tilde{t}$  because we like to work with symmetric operators, and we want a  $t$  with the dimension of a length.

Thus,

$$-i\lambda^{-1}t = \sum_{\mu} \frac{d\eta_{\mu}^{(\tau)}}{d\tau} \Big|_{\tau=0} \frac{\partial}{\partial \eta_{\mu}} = \eta_3 \partial_0 + \eta_2 \partial_1 + \eta_1 \partial_2 + \eta_0 \partial_3 .$$

On the same line, one can compute  $x$  and  $y$ . The full list of vector fields is:

$$\begin{aligned} t &= i\lambda(\eta_3\partial_0 + \eta_2\partial_1 + \eta_1\partial_2 + \eta_0\partial_3) , \\ x &= i\lambda(-\eta_2\partial_0 + \eta_3\partial_1 - \eta_0\partial_2 + \eta_1\partial_3) , \\ y &= i\lambda(\eta_1\partial_0 - \eta_0\partial_1 + \eta_3\partial_2 - \eta_2\partial_3) , \end{aligned}$$

and the commutation rules are those of “ $i \cdot sl(2, \mathbb{R})$ ”:

$$[t, x] = 2i\lambda y , \quad [t, y] = 2i\lambda x , \quad [x, y] = -2i\lambda t .$$

In contrast with the two dimensional Snyder’s model, now  $\{t, x, y\}$  is the basis of a Lie algebra. Furthermore, this algebra has the correct number of generators to represent a three-dimensional noncommutative space, dual to the three dimensional deSitter space.

This idea can easily be generalized: starting with a Lie group  $M$ , one can consider  $U(\text{Lie } M)$  as noncommutative space ‘dual’ to  $M$ , and eventually study it with tools of noncommutative geometry.

A celebrated (compact, Riemannian) example is the *fuzzy sphere* [11], quotient of  $U(su(2))$  for the ideal generated by  $J^2 - c$ , with  $J^2$  the Casimir and  $c$  a suitable constant.

In the next section we consider a simple, two-dimensional example.

#### IV. A TWO-DIMENSIONAL MODEL: $\kappa$ -MINKOWSKI

In two dimensions, the unique (real connected) non-abelian Lie group is the matrix group of elements [12, sec. 10.1]

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} , \quad (a, b) \in \mathbb{R}^+ \times \mathbb{R} .$$

That is, (the connected component of) the group of affine transformations of the real line:

$$\begin{pmatrix} y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} ay + b \\ 1 \end{pmatrix} , \quad y \in \mathbb{R} .$$

The map

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{pmatrix} ,$$

gives an isomorphism with  $Sb(2, \mathbb{R})$ , the group of special upper-triangular real matrices, so the variety  $M \simeq Sb(2, \mathbb{R})$  is the starting point.

We want to compute the vector fields and identify them with “noncommutative coordinates”. Let  $L = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$  be the generic element of the Lie algebra,  $a, b \in \mathbb{R}$ , and  $-i\tilde{L}$  the associated vector field:

$$(-i\tilde{L}f)(m) = \left. \frac{d}{d\tau} \right|_{\tau=0} f(\exp\{\tau L\}m) , \quad f \in C^\infty(M) .$$

We use physicists habit of working with selfadjoint operators. Since

$$L^n = \begin{pmatrix} a^n & a^{n-1}b \\ 0 & 1 \end{pmatrix} , \quad \exp\{\tau L\} = \begin{pmatrix} e^{\tau a} & \frac{e^{\tau a}-1}{a}b \\ 0 & 1 \end{pmatrix} ,$$

if we indicate with  $m = \begin{pmatrix} \eta_0 & \eta_1 \\ 0 & 1 \end{pmatrix}$  the generic point of  $M$ ,  $(\eta_0, \eta_1) \in \mathbb{R}^+ \times \mathbb{R}$ , it is easy to compute

$$(\tilde{L}f)(\eta_0, \eta_1) = i(a\eta_0\partial_0 + (a\eta_1 + b)\partial_1)f(\eta_0, \eta_1) ,$$

with  $\partial_\mu = \partial/\partial\eta_\mu$ . We fix the basis

$$x = i\lambda\partial_1 , \quad t = i\lambda(\eta_0\partial_0 + \eta_1\partial_1) ,$$

for vector fields on  $M$ , and define the algebra of “functions on the noncommutative space-time” to be the algebra of polynomials generated by  $x$  and  $t$ . The presence of the length  $\lambda$  guarantees the correct physical dimensions, and enables us to recover  $\mathbb{R}^2$  as  $\lambda \rightarrow 0$  limit.

Thus, the algebra is generated by  $x$  and  $t$  modulo

$$[x, t] = i\lambda x . \quad (7)$$

This is just  $U(sb(2, \mathbb{R}))$ , the two-dimensional version of the so-called  $\kappa$ -Minkowski, introduced in [2] as homogeneous space for  $\kappa$ -Poincaré [13].

The same Lie-algebra, but with different real structure, emerges in the Weyl quantization of  $S^1 \times \mathbb{R}$ . In that case  $x$  is unitary, thus the space is “compact” in the  $x$ -direction.

Now, let  $\mu$  be the (left) Haar measure on  $M$ . Explicitly, for an integrable function  $f$  on  $M$ :

$$\int_M f d\mu = \int_{\mathbb{R}^+ \times \mathbb{R}} f(\eta_0, \eta_1) \eta_0^{-1} d\eta_0 d\eta_1 .$$

You can easily verify the invariance with respect to the left regular action, i.e.  $\int_M f d\mu = \int_M f' d\mu$  with  $f'(\eta_0, \eta_1) = f(a\eta_0, a\eta_1 + b)$  and for all  $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ .

With this measure, we can define the Hilbert space  $\mathcal{H} = \mathcal{L}^2(M, \mu)$  with inner product  $\langle \varphi, \psi \rangle = \int_M \varphi^* \psi d\mu$ .

The measure being invariant, it means that ‘finite’ transformations of the group  $M$  act as isometries on the associated Hilbert space, i.e. as unitary operators. Thus, the vector fields  $x$  and  $t$ , the generators of these transformations, are represented by (unbounded) selfadjoint linear operators. I mean:

$$\begin{aligned} \langle x^* \varphi, \psi \rangle &:= \langle \varphi, x\psi \rangle \equiv \langle x\varphi, \psi \rangle , \\ \langle t^* \varphi, \psi \rangle &:= \langle \varphi, t\psi \rangle \equiv \langle t\varphi, \psi \rangle , \end{aligned}$$

for all  $\varphi, \psi \in \mathcal{H}$  and in the domain of  $x$ , resp.  $t$  (it is an easy check to verify these equations). The self-adjointness of  $x$  and  $t$  allows us to interpret them as quantum-mechanical observables.

### A. $\kappa$ -Minkowski more in depth

Following Snyder’s idea, we interpret vector fields on  $M$  as “coordinates”, and define “momenta” as the following functions in  $C(M)$ :

$$P = \lambda^{-1} \eta_1 , \quad E = \lambda^{-1} \log \eta_0 .$$

Again, the presence of  $\lambda$  is for dimensional reasons and we choose  $E$  as a logarithm because we want it in  $\mathbb{R}$ , and not in  $\mathbb{R}^+$ .

The commutators defining phase-space are

$$[x, E] = 0 , \quad [x, P] = i = [t, E] , \quad [t, P] = i\lambda P , \quad (8)$$

and reduce to the classical Heisenberg algebra for  $\lambda = 0$  (physically, this can be interpreted as a “low-energy limit”, i.e. as an approximation for  $|\lambda E|, |\lambda P| \ll 1$ ).

To complete the picture, we want to define the analogue of a boost generator: the generator of a transformation of  $M$  with a fixed point (of course, we choose the unit element  $(\eta_0, \eta_1) = (1, 0)$  as fixed point).

If we define

$$iN = E\partial_P + P\partial_E \equiv \eta_0\eta_1\partial_0 + \log \eta_0\partial_1 \quad ,$$

$N$  acts on  $E$  and  $P$  as a classical boost, and  $\{E, P, N\}$  generate the (undeformed) Poincaré algebra:

$$[N, E] = -iP \quad , \quad [N, P] = -iE \quad .$$

Since  $N \equiv -Pt - (E - \lambda P^2)x$ , the action on coordinates is

$$[N, x] = i(t - \lambda Px) \quad , \quad [N, t] = i(1 - \lambda E - \lambda^2 P^2)x + i\lambda Pt \quad .$$

Thus, the action on coordinates is non-linear and reduces to a classical boost for  $\lambda \rightarrow 0$ .

Like in the Snyder’s case, we have deformed  $\mathbb{R}^2$  into a noncommutative space without breaking the Poincaré symmetry.

In the Snyder case, the coordinates are vector fields on  $SO(2, 1)$  orthogonal to the submanifold  $SO(1, 1)$ , they are a  $G$ -algebra module for Poincaré and don’t close an algebra.

In the  $\kappa$ -Minkowski case, the Lie group associated to  $\{E, P, N\}$  is Poincaré, isomorphic to  $SO(1, 1) \ltimes Sb(2, \mathbb{R})$ , where  $SO(1, 1) \simeq \mathbb{R}$  is the subgroup generated by  $N$ . Coordinates are vector fields on  $M = Sb(2, \mathbb{R}) \simeq \{SO(1, 1) \ltimes Sb(2, \mathbb{R})\}/SO(1, 1)$ , and close a Lie algebra isomorphic to “ $i \cdot sb(2, \mathbb{R})$ ”. The boosts  $SO(1, 1)$  are the isotropy transformations of  $M$ , that we identify with momentum space. The Poincaré symmetry of spacetime is obtained by dualizing the action of  $SO(1, 1)$  and taking the cross-product with momenta.

All what done in this subsection can be generalized to  $n + 1$  dimensions. Just substitute  $M$  with the matrix group of elements

$$\begin{pmatrix} e^a & b \\ 0 & \mathbb{1} \end{pmatrix} \quad , \quad \text{with } a \in \mathbb{R}, b \in \mathbb{R}^n, \text{ and } \mathbb{1} \text{ the } n \times n \text{ identity matrix.}$$

The associated Lie algebra is again called  $\kappa$ -Minkowski [2] and arises in quantum group theory as a quantum homogeneous space.

## B. Quantum group of symmetries for $\kappa$ -Minkowski

One could object that the action of the boost  $N$  on  $\kappa$ -Minkowski coordinates depends on momenta. In more mathematical terms: spacetime is not an algebra module for Poincaré, although phase-space is.

It is obvious that a noncommutative space cannot be a quantum homogeneous space for a Lie group (I mean ‘embeddable’, see appendix A). Or, in other words, that the algebra of functions on a Lie group has only commutative subgroups.

It results that  $\kappa$ -Minkowski is embeddable in a quantum group,  $\kappa$ -Poincaré, whose  $\lambda \rightarrow 0$  limit is the classical Poincaré group. This fact will be briefly discussed in the present subsection.

We work first in the Euclidean case, and then pass to the pseudo-Euclidean version through a Wick rotation.



Let  $z = x + it$ . The algebra of coordinates is generated by  $z$  and  $z^*$ , with relation

$$[z, z^*] = \lambda(z + z^*) . \quad (9)$$

We write the coaction of a yet-unidentified Hopf algebra, generated by  $\{w, w^*, a, a^*\}$ , as

$$\Delta_L(z) = w \otimes z + a \otimes 1 .$$

This extends to a  $*$ -morphism if it is compatible with the commutator (9). Using the rule  $[A \otimes B, C \otimes D] = [A, C] \otimes BD + CA \otimes [B, D]$ , we write this condition as

$$\begin{aligned} [\Delta(z), \Delta(z^*)] &= [w, w^*] \otimes zz^* + (\lambda w^* w + [w, a^*]) \otimes z^* + (\lambda w^* w + [w, a^*])^* \otimes z + \\ &+ [a^*, a] \otimes 1 \equiv \lambda(w \otimes z + w^* \otimes z^* + (a + a^*) \otimes 1) = \Delta([z, z^*]) . \end{aligned}$$

The commutator (9) is covariant if and only if

$$[w, w^*] = 0 , \quad [a^*, a] = \lambda(a + a^*) , \quad (10a)$$

$$[w, a^*] = \lambda(1 - w^*)w , \quad [w^*, a] = -\lambda w^*(1 - w) . \quad (10b)$$

Notice that the map  $z \mapsto a$  extends to a  $*$ -algebra morphism between  $\kappa$ -Minkowski and a subalgebra of the symmetry Hopf-algebra. This means that  *$\kappa$ -Minkowski is an embeddable homogeneous space* for a suitable Hopf algebra that we are going to construct. The further condition  $w^*w = 1$  ensures the existence of the antipode and of a quadratic invariant (i.e.  $z^*z = x(x + \lambda) + t^2$ ). The last two conditions in (10) are one the conjugate of the other, and both are equivalent to the condition  $[w, a] = w[w^*w, a] - w[w^*, a]w \equiv -w[w^*, a]w = \lambda(1 - w)w$ .

To conclude, the Hopf algebra of symmetries is generated by  $w, a$  and their conjugates, with relations:

$$\begin{aligned} w^*w &= ww^* = 1 , \\ [a^*, a] &= \lambda(a + a^*) , \\ [w, a] &= \lambda(1 - w)w . \end{aligned}$$

The coproduct, counit and antipode are the standard ones for a matrix quantum group, given by the well-looking formulas:

$$\begin{aligned} \Delta \begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix} , \\ \epsilon \begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \\ S \begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix}^{-1} \equiv \begin{pmatrix} w^* & -w^*a \\ 0 & 1 \end{pmatrix} , \end{aligned}$$

and telling us that  $\begin{pmatrix} w & a \\ 0 & 1 \end{pmatrix}$  is a corepresentation (the fundamental one).

This Hopf algebra is the bicrossproduct [14] of a classical rotation subgroup  $U(1)$  and  $U(sb(2, \mathbb{R}))$ . The dual Hopf algebra is generated by three real elements  $\{N, E, P\}$  (for notational convenience we continue to use the same symbols of sec. IV A, but we remark that the Hopf algebra is different).

The  $\{E, P\}$  subalgebra is dual to  $\kappa$ -Minkowski, and can be determined (in an equivalent manner) as follows. Since  $\kappa$ -Minkowski is a classical Lie-algebra, the dual is the coordinate Hopf algebra of the group of elements  $e^{ip_1x}e^{ip_0t}$ , where  $(p_0, p_1) \in \mathbb{R}^2$  and  $e := \exp$  being the exponential map (this is the connected component  $M$  of the group of affine transformations of  $\mathbb{R}$ , just the group which we started from). We define  $E$  and  $P$  to be the following functions:

$$E(e^{ip_1x}e^{ip_0t}) = p_0 \quad , \quad P(e^{ip_1x}e^{ip_0t}) = p_1 \quad .$$

Of course,  $[E, P] = 0$ . The product rule between elements of the group (given by the Baker-Campbell-Hausdorff formula and providing an obvious isomorphism with  $M$ ):

$$(e^{ip_1x}e^{ip_0t})(e^{ip'_1x}e^{ip'_0t}) = e^{i(p_1+p'_1e^{\lambda p_0})x}e^{i(p_0+p'_0)t} \quad ,$$

tells us that the coproduct, dual to the product, is determined by the equations

$$\begin{aligned} (\Delta E)(e^{ip_1x}e^{ip_0t}, e^{ip'_1x}e^{ip'_0t}) &= E(e^{i(p_1+p'_1e^{\lambda p_0})x}e^{i(p_0+p'_0)t}) = p_0 + p'_0 \quad , \\ (\Delta P)(e^{ip_1x}e^{ip_0t}, e^{ip'_1x}e^{ip'_0t}) &= P(e^{i(p_1+p'_1e^{\lambda p_0})x}e^{i(p_0+p'_0)t}) = p_1 + p'_1e^{\lambda p_0} \quad . \end{aligned}$$

Then, one has:

$$\begin{aligned} \Delta E &= E \otimes 1 + 1 \otimes E \quad , \\ \Delta P &= P \otimes 1 + e^{\lambda E} \otimes P \quad . \end{aligned}$$

Similarly, dualizing unit and inverse, one calculate counit and antipode

$$\epsilon(E) = \epsilon(P) = 0 \quad , \quad S(E) = -E \quad , \quad S(P) = -Pe^{-\lambda E} \quad .$$

The full Hopf-algebra of symmetries is the bicrossproduct dual to  $\mathbb{C}[U(1)] \rhd\blacktriangleleft U(\mathfrak{sb}(2, \mathbb{R}))$ . Explicitly, the commutators are:

$$[N, P] = \frac{i}{2\lambda}(1 - e^{2\lambda E}) + \frac{i\lambda}{2}P^2 \quad , \quad [N, E] = iP \quad ,$$

while  $\epsilon(N) = 0$ ,  $S(N) = -e^{-\lambda E}N$  and

$$\Delta(N) = N \otimes 1 + e^{\lambda E} \otimes N \quad .$$

Notice that in the basis

$$P_x = E \quad , \quad P_y = Pe^{-\lambda E/2} \quad , \quad J = Ne^{-\lambda E/2} \quad ,$$

the Hopf-algebra is just the *quantum Euclidean group* derived in [15] by contraction of  $U_q(\mathfrak{su}(2))$ . The pseudo-Euclidean version is the Hopf algebra generated by  $\{N', E', P'\}$  with relations:

$$[E', P'] = 0 \quad , \quad [N', P'] = \frac{i}{2\lambda}(1 - e^{2\lambda E'}) - \frac{i\lambda}{2}P'^2 \quad , \quad [N', E'] = -iP' \quad ,$$

and with the same coproduct as in the Euclidean case. This is the two-dimensional version of  $\kappa$ -Poincaré, obtained for the first time in [13] by contraction of  $U_q(\mathfrak{so}(3, 2))$ .

Dualizing the coaction and passing to the pseudo-Euclidean signature, we find that the action is standard:

$$\begin{aligned} N' \triangleright x &= it \quad , & P' \triangleright x &= -i \quad , & E' \triangleright x &= 0 \quad , \\ N' \triangleright t &= ix \quad , & P' \triangleright t &= 0 \quad , & E' \triangleright t &= -i \quad . \end{aligned}$$

The action is undeformed, but the coproduct is. For this reason, one can calculate

$$\begin{aligned} N' \triangleright x^2 &= (N' \triangleright x)x + x(N' \triangleright x) = i(tx + xt) , \\ N' \triangleright t^2 &= (N' \triangleright t)t + (e^{\lambda E'} \triangleright t)(N' \triangleright t) = i(tx + xt) + \lambda x = N' \triangleright x^2 - i\lambda N' \triangleright t , \end{aligned}$$

and prove that the quadratic invariant is deformed into  $x^2 - t(t + i\lambda)$ , that is:

$$N' \triangleright \{x^2 - t(t + i\lambda)\} = 0 .$$

Finally, a phase-space can be obtained as a cross-product of momenta and coordinates. The relations defining the phase-space are

$$fg = (f_{(1)} \triangleright g)f_{(2)} ,$$

for  $f$  a generic function of the momenta,  $g$  a function of the coordinates and  $\Delta T = T_{(1)} \otimes T_{(2)}$  the Sweedler notation [14]. Thus:

$$[x, P'] = [t, E'] = i , \quad [t, P'] = i\lambda P' , \quad [x, E'] = 0 .$$

This is the same algebra defined by eq. (8).

A second (different) phase-space can be constructed via a cross-product using the right canonical action of momenta on coordinates, instead of the left one.

The construction of phase space as a cross-product (in  $3 + 1$  dimensions) was performed in [16, 17], an analysis of physical consequences is in [18].

## V. SPECTRAL GEOMETRY OF $\kappa$ -MINKOWSKI

Let us quote from [7] the definition of spectral triple, noncommutative generalization of the notion of Riemannian spin<sup>c</sup> manifold.

**Definition 1** *A spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is given by an involutive algebra  $\mathcal{A}$ , a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  by bounded operators on a Hilbert space  $\mathcal{H}$ , and a self-adjoint operator  $D = D^*$  with dense domain in  $\mathcal{H}$ , such that:*

(S1)  $\pi(a)(D^2 + 1)^{-1/2}$  is a compact operator for all  $a \in \mathcal{A}$ ;

(S2)  $[D, \pi(a)]$  is a bounded operator for all  $a \in \mathcal{A}$ .

A commutative example is the canonical spectral triple associated to the spin structure of  $\mathbb{R}^2$ :  $(C_0^\infty(\mathbb{R}^2), \mathcal{H}, \not{D})$ , with  $C_0^\infty(\mathbb{R}^2)$  smooth functions vanishing at infinity on  $\mathbb{R}^2$ ,  $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$  the Hilbert space of  $L^2$ -spinors and  $\not{D}$  the Dirac operator:

$$\not{D} = i(\sigma_1 \partial_0 + \sigma_2 \partial_1) = \begin{pmatrix} 0 & i\partial_0 + \partial_1 \\ i\partial_0 - \partial_1 & 0 \end{pmatrix} .$$

Here  $\sigma_j$  are the Pauli matrices,  $\eta_\mu$  coordinates in  $\mathbb{R}^2$  and  $\partial_\mu := \partial/\partial\eta_\mu$  the corresponding derivatives. Greek letters run over 0, 1. We define also  $\vec{\eta} = (\eta_0, \eta_1)$ .

$C_0^\infty(\mathbb{R}^2)$  is a Fréchet pre- $C^*$ -algebra. The supremum norm is equivalent to the operator norm on  $\mathcal{H}$ , and the  $C^*$ -algebra completion is  $C_0(\mathbb{R}^2)$ , the algebra of continuous functions vanishing at infinity.

**Definition 2** A spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is even if exists a grading  $\gamma \in \mathcal{B}(\mathcal{H})$ ,  $\gamma = \gamma^*$  and  $\gamma^2 = 1$ , such that  $\gamma D = -D\gamma$  and  $a\gamma = \gamma a \forall a \in \mathcal{A}$ .

The operator:

$$\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (11)$$

is a grading for the canonical spectral triple on  $\mathbb{R}^2$ .

We quote also the notion of equivariance with respect to the action of a Lie group.

**Definition 3** Let  $H$  be Lie group,  $\rho : H \rightarrow \mathcal{B}(\mathcal{H})$  a representation and  $\triangleright : H \times \mathcal{A} \rightarrow \mathcal{A}$  a covariant action. An even spectral triple  $(\mathcal{A}, \mathcal{H}, D, \gamma)$  is  $H$ -equivariant if:

- (E1)  $\rho(h)\pi(a)\rho(h)^{-1} = \pi(h \triangleright a)$  for all  $a \in \mathcal{A}$ ,  $h \in H$ ;
- (E2)  $\rho(h)D\rho(h)^{-1} = D$  and  $\rho(h)\gamma\rho(h)^{-1} = \gamma$  for all  $h \in H$ .

The spin structure of  $\mathbb{R}^2$  is equivariant with respect to the spin representation of  $H := ISO(2) \simeq SO(2) \ltimes \mathbb{R}^2$ , the group of isometries of the Euclidean plane  $\mathbb{R}^2$ .

If  $(R, \vec{v}), (R', \vec{v}') \in SO(2) \times \mathbb{R}^2$ , the multiplication law of  $ISO(2)$  is:

$$(R, \vec{v}) \cdot (R', \vec{v}') = (RR', \vec{v} + R\vec{v}'),$$

and the spin representation  $\rho : H \rightarrow \mathcal{B}(\mathcal{H})$  is defined by:

$$\{\rho(R, \vec{v})\psi\}(\vec{\eta}) := R \cdot \psi(R^{-1}(\vec{\eta} - \vec{v})), \quad \forall \psi \in \mathcal{H}. \quad (12)$$

The representation  $\pi$  of the algebra satisfies (E1) if we define  $\triangleright$  to be the pull-back of the natural action on  $\mathbb{R}^3$ :

$$\{(R, \vec{v}) \triangleright f\}(\vec{\eta}) = f(R^{-1}(\vec{\eta} - \vec{v})), \quad \forall f \in C_0^\infty(\mathbb{R}^2).$$

Differentiating the action of the group, we arrive at the equivalent notion of  $U(\text{Lie}H)$ -equivariance. This notion can be generalized to a generic Hopf-algebra.

**Definition 4** Let  $\mathcal{U}$  be an Hopf-algebra,  $\rho : \mathcal{U} \rightarrow \mathcal{B}(\mathcal{H})$  a representation and  $\triangleright : \mathcal{U} \times \mathcal{A} \rightarrow \mathcal{A}$  a covariant action. An even spectral triple  $(\mathcal{A}, \mathcal{H}, D, \gamma)$  is  $\mathcal{U}$ -equivariant if:

- (E1')  $\rho(u_{(1)})\pi(a)\rho(Su_{(2)}) = \pi(u \triangleright a)$  for all  $a \in \mathcal{A}$ ,  $u \in \mathcal{U}$ ;
- (E2')  $\rho(u)D = D\rho(u)$  and  $\rho(u)\gamma = \gamma\rho(u)$  for all  $u \in \mathcal{U}$ .

In the following we construct the algebras replacing continuous and smooth functions vanishing at infinity associated to  $\kappa$ -Minkowski, and describe the spinor representation.

Since the metric properties of the space depend on the Dirac operator, at this point there is no difference between Euclidean and Lorentzian version.

We then analyze the problem of constructing an Euclidean Dirac operator. We exhibit an operator that is essentially the Dirac operator on the commutative subspace  $\mathbb{R}$  of  $\kappa$ -Minkowski, and prove that it defines a spectral triple. We search also for an equivariant Dirac operator and find that there is just one, and does not satisfy the axioms of a spectral triple.

### A. The algebra of polynomials of the “noncommutative coordinates”

Let us recall the construction of the algebra of coordinates on  $\kappa$ -Minkowski.

We call  $G := (\mathbb{R}^2, \dot{+})$  the space  $\mathbb{R}^2$  with deformed sum:

$$(p_0, p_1) \dot{+} (p'_0, p'_1) = (p_0 + p'_0, p_1 + p'_1 e^{\lambda p_0}) \quad \forall (p_0, p_1), (p'_0, p'_1) \in \mathbb{R}^2 .$$

The map:

$$(p_0, p_1) \mapsto g := \begin{pmatrix} e^{\lambda p_0} & p_1 \\ 0 & 1 \end{pmatrix} ,$$

is an isomorphism between  $G$  and  $\text{Aff}_0(\mathbb{R})$ , (the connected component of) the group of affine transformation of  $\mathbb{R}$ . We will identify  $G$  and  $\text{Aff}_0(\mathbb{R})$ .

There is only one unitary irreducible infinite-dimensional representation of the group  $\text{Aff}(\mathbb{R})$ , defined on the Hilbert space  $L^2(\mathbb{R}^*, d\eta_1/|\eta_1|)$  by [19]:

$$\{g \cdot \varphi\}(\eta_1) := e^{ib\eta_1} \varphi(a\eta_1) \quad , \quad \varphi \in L^2(\mathbb{R}^*, d\eta_1/|\eta_1|) , \quad g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{Aff}(\mathbb{R}).$$

We take the tensor product of this representation with the trivial representation  $g \mapsto e^{i\eta_0 p_0}$  of the abelian subgroup  $\mathbb{R}$ . Then we take the direct integral on  $\eta_0 \in \mathbb{R}$  with an arbitrary measure  $d\mu(\eta_0)$  on  $\mathbb{R}$ .

The result is a unitary representation of  $G$  on the space:

$$\mathcal{H}_\mu := L^2(\mathbb{R} \times \mathbb{R}^*, d\mu(\eta_0) |\eta_1|^{-1} d\eta_1) \otimes \mathbb{C}^2 , \quad (13)$$

defined by:

$$\begin{aligned} \pi : G &\rightarrow \mathcal{B}(\mathcal{H}_\mu) \quad , \quad \vec{p} \mapsto \pi(\vec{p}) \quad , \\ \{\pi(\vec{p})\psi\}(\vec{\eta}) &= e^{i\vec{p} \cdot \vec{\eta}} \psi(\eta_0, \eta_1 e^{\lambda p_0}) \quad , \end{aligned} \quad (14)$$

where  $\psi \in \mathcal{H}_\mu$ . One can explicitly check that it is an homomorphism:

$$\pi(\vec{p})\pi(\vec{p}') = \pi(p_0 + p'_0, p_1 + p'_1 e^{\lambda p_0}) .$$

We indicate with  $\mathcal{H}_0$  the Hilbert space obtained taking as  $d\mu(\eta_0)$  the Lebesgue measure  $d\eta_0$ .

In the previous section, we have defined the “noncommutative coordinates” on  $\kappa$ -Minkowski as the vector fields associated to the generators of  $U(\mathfrak{g})$ :

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} , \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

where  $\mathfrak{g}$  is the Lie algebra of  $G$ .

We can obtain a representation of these coordinates on  $\mathcal{H}_\mu$  using the differential of  $\pi$ :

$$\begin{aligned} \hat{x}_0 &= i d\pi \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} = \eta_0 - i\lambda\eta_1 \partial_1 \quad , \\ \hat{x}_1 &= i d\pi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \eta_1 \quad . \end{aligned}$$

The operators  $(\hat{x}_0, \hat{x}_1)$  generate an algebra isomorphic to the  $\kappa$ -Minkowski algebra:

$$[\hat{x}_0, \hat{x}_1] = -i\lambda\hat{x}_1 . \quad (15)$$

We indicate with  $\mathbb{R}_\lambda^2$  the virtual “quantum space” associated to the algebra  $U(\mathfrak{g})$ , and call elements of  $U(\mathfrak{g})$  the ‘polynomial functions’ on  $\kappa$ -Minkowski space.

When  $\lambda = 0$ , the representation  $d\pi$  reduces to the ordinary (unbounded) representation of polynomial functions on  $\mathbb{R}^2$  via pointwise multiplication on  $\mathcal{H}_\mu$ .

### B. Continuous and smooth ‘functions’ on $\kappa$ -Minkowski space

To construct a spectral triple with the polynomial algebra  $U(\mathfrak{g})$  is problematic, since  $\hat{x}_\mu$  cannot be represented by bounded operators, and (15) can be satisfied only on a dense domain in  $\mathcal{H}_\mu$ . It is the same problem one encounters in the canonical quantization of phase-space in quantum-mechanics. A possible solution is to shift the attention from  $\hat{x}_\mu$  to complex exponentials, that is, from the Lie algebra  $\mathfrak{g}$  to the Lie group  $G$ . This is Weyl quantization, defined as a map associating complex exponentials on  $\mathbb{R}^2$  to elements of  $G$ , represented by unitary operators on  $\mathcal{H}_\mu$ . The quantization map can be extended to an involutive subalgebra of  $C_0(\mathbb{R}^2)$  using Fourier transform.

We call  $\text{Fun}(\mathbb{R}^2)$  the following class of functions:

$$\text{Fun}(\mathbb{R}^2) := C_0(\mathbb{R}^2) \cap \mathcal{H}_0 ,$$

and define the following quantization map:

$$\begin{aligned} \Omega : \text{Fun}(\mathbb{R}^2) &\rightarrow \mathcal{B}(\mathcal{H}_\mu) , \\ f &\mapsto \Omega(f) = \int_{\mathbb{R}^2} d^2p \tilde{f}(\vec{p}) \pi(\vec{p}) , \end{aligned}$$

where  $\pi$  is the representation (14) and  $\tilde{f}$  is the Fourier transform of  $f$ :

$$\tilde{f}(\vec{p}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\vec{\eta}) e^{-i\vec{p}\vec{\eta}} d^2\eta .$$

Although  $f = e^{i\vec{p}\vec{\eta}} \notin C_0(\mathbb{R}^2)$ , one can formally verify taking  $\tilde{f}(\vec{p}') = \delta^{(2)}(\vec{p} - \vec{p}')$  that:

$$\Omega(e^{i\vec{p}\vec{\eta}}) = \pi(\vec{p}) ,$$

and since  $\eta_\mu = -i \frac{\partial}{\partial p_\mu} \Big|_{\vec{p}=0} e^{i\vec{p}\vec{\eta}}$ ,

$$\Omega(\eta_\mu) = -i \frac{\partial}{\partial p_\mu} \Big|_{\vec{p}=0} \pi(\vec{p}) \equiv \hat{x}_\mu .$$

The explicit expression of the quantization map is:

$$\{\Omega(f)\psi\}(\vec{\eta}) = \int_{\mathbb{R}^2} \tilde{f}(\vec{p}) e^{i\vec{p}\vec{\eta}} \psi(\eta_0, \eta_1 e^{\lambda p_0}) d^2p , \quad f \in \text{Fun}(\mathbb{R}^2) , \quad \psi \in \mathcal{H}_\mu , \quad (16)$$

and we need to verify that it defines bounded operator.

**Proposition 5** *The quantization map  $\Omega$ , defined by (16), sends  $\text{Fun}(\mathbb{R}^2)$  into bounded operators  $\mathcal{B}(\mathcal{H}_\mu)$ . For this class of functions, the operator norm is bounded by:*

$$\|\Omega(f)\|^2 := \sup_{\psi \in \mathcal{H}_\mu, \psi \neq 0} \frac{\|\Omega(f)\psi\|_{\mathcal{H}_\mu}^2}{\|\psi\|_{\mathcal{H}_\mu}^2} \leq \frac{1}{2\pi} \|f\|_{\mathcal{H}_0}^2 , \quad (17)$$

where  $\|\cdot\|_{\mathcal{H}_\mu}$  indicate the norm in the Hilbert space  $\mathcal{H}_\mu$ .

*Proof:*

Let  $a = e^{\lambda p_0}$  and call:

$$F(\vec{\eta}, a) = \frac{1}{2\pi} \int_{\mathbb{R}} d\eta'_0 f(\eta'_0, \eta_1) e^{ip_0(\eta_0 - \eta'_0)} = \int_{\mathbb{R}} d\eta_1 \tilde{f}(\vec{p}) e^{i\vec{p}\vec{\eta}} .$$

Then:

$$\{\Omega(f)\psi\}(\vec{\eta}) = \int_{\mathbb{R}} dp_0 F(\vec{\eta}, a) \psi(\eta_0, a\eta_1) = \int_{\mathbb{R}^+} \frac{da}{\lambda a} F(\vec{\eta}, a) \psi(\eta_0, a\eta_1) .$$

Since (partial) Fourier transform is an isometry of  $L^2$ , then  $F \in L^2(\mathbb{R}, dp_0) = L^2(\mathbb{R}^+, da/\lambda a)$  for each fixed  $\vec{\eta}$ . Using Schwartz inequality:

$$\begin{aligned} |\Omega(f)\psi(\vec{\eta})|^2 &\leq \left( \int_{\mathbb{R}^+} \frac{da}{\lambda a} |F(\vec{\eta}, a) \psi(\eta_0, a\eta_1)| \right)^2 \\ &\leq \left( \int_{\mathbb{R}^+} \frac{da}{\lambda a} |F(\vec{\eta}, a)|^2 \right) \left( \int_{\mathbb{R}^+} \frac{da}{a} |\psi(\eta_0, a\eta_1)|^2 \right) . \end{aligned} \quad (18)$$

Now:

$$\begin{aligned} \int_{\mathbb{R}^+} \frac{da}{\lambda a} |F(\vec{\eta}, a)|^2 &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^+} \frac{da}{\lambda a} \left| \int_{\mathbb{R}} d\eta'_0 f(\eta'_0, \eta_1) e^{ip_0(\eta_0 - \eta'_0)} \right|^2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} dp_0 \int_{\mathbb{R}^2} d\eta'_0 d\eta''_0 \bar{f}(\eta'_0, \eta_1) f(\eta''_0, \eta_1) e^{ip_0(\eta'_0 - \eta''_0)} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d\eta'_0 |f(\eta'_0, \eta_1)|^2 , \end{aligned} \quad (19a)$$

$$\int_{\mathbb{R}^+} \frac{da}{a} |\psi(\eta_0, a\eta_1)|^2 \leq \int_{\mathbb{R}^*} \frac{da}{|a|} |\psi(\eta_0, a\eta_1)|^2 = \int_{\mathbb{R}^*} \frac{d\eta'_1}{|\eta'_1|} |\psi(\eta_0, \eta'_1)|^2 , \quad (19b)$$

where  $\eta'_1 = a\eta_1$  and in the last step we used dilatation invariance of the measure.

Using the inequalities (18) and (19) we arrive at:

$$\begin{aligned} \|\Omega(f)\psi\|_{\mathcal{H}_\mu}^2 &= \int_{\mathbb{R} \times \mathbb{R}^*} \frac{d\mu(\eta_0) d\eta_1}{|\eta_1|} |\Omega(f)\psi(\vec{\eta})|^2 \\ &\leq \frac{1}{2\pi} \left( \int_{\mathbb{R} \times \mathbb{R}^*} \frac{d\eta'_0 d\eta_1}{|\eta_1|} |f(\eta'_0, \eta_1)|^2 \right) \left( \int_{\mathbb{R} \times \mathbb{R}^*} \frac{d\mu(\eta_0) d\eta'_1}{|\eta'_1|} |\psi(\eta_0, \eta'_1)|^2 \right) \\ &= \frac{1}{2\pi} \|f\|_{\mathcal{H}_0}^2 \cdot \|\psi\|_{\mathcal{H}_\mu}^2 . \end{aligned}$$

Using the last inequality, we find the upper bound (17) for the operator norm.  $\square$

We define the following  $*$ -product on  $\text{Fun}(\mathbb{R}^2)$ :

$$f_1 * f_2 := \Omega(f_1) f_2 , \quad \forall f_{1,2} \in \text{Fun}(\mathbb{R}^2) . \quad (20)$$

Let  $\text{Fun}^\infty(\mathbb{R}^2)$  be the subspace of smooth functions of  $\text{Fun}(\mathbb{R}^2)$ .

**Proposition 6**  $\mathcal{A} := (\text{Fun}(\mathbb{R}^2), *)$  and  $\mathcal{A}^\infty := (\text{Fun}^\infty(\mathbb{R}^2), *) \subset \mathcal{A}$  are involutive algebras (associative without unit).  $\Omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\mu)$  is a unitary representation.

*Proof:*

Since  $\Omega(f_1)$  is a bounded operator on  $\mathcal{H}_\mu$ , for all  $\mu$ , and  $f_2 \in \mathcal{H}_0$ , the product  $f_1 * f_2$  is in  $\mathcal{H}_0$ . Moreover,  $f_1 * f_2 \in C_0(\mathbb{R}^2)$  and is smooth if  $f_1$  and  $f_2$  are.

By construction:

$$\Omega(f_1 * f_2) = \Omega(f_1)\Omega(f_2) .$$

This guarantees associativity of the product, and proves that  $\Omega$  is a representation.

Finally,  $\Omega(\bar{f}_1) = \Omega(f_1)^+$  is the adjoint of  $\Omega(f_1)$ , since the representation  $\pi$  of the group  $\text{Aff}_0(\mathbb{R})$  is unitary. So we have a unitary representation of  $\mathcal{A}$ .  $\square$

We call  $\mathcal{A}$  the algebra of ‘continuous functions’ on  $\kappa$ -Minkowski space, and  $\mathcal{A}^\infty$  the subalgebra of ‘smooth functions’. Since  $1 \notin \mathcal{A}$ , the space is not compact.

From (16), using Leibniz rule and the property  $ip_\mu \tilde{f} = \partial_\mu \tilde{f}$ , we deduce the following ‘deformed’ Leibniz rule:

$$\partial_0\{\Omega(f_1)f_2\} = \Omega(\partial_0 f_1)f_2 + \Omega(f_1)\partial_0 f_2 , \quad (21a)$$

$$\partial_1\{\Omega(f_1)f_2\} = \Omega(\partial_1 f_1)f_2 + \Omega(e^{-i\lambda\partial_0} f_1)\partial_1 f_2 , \quad (21b)$$

or equivalently:

$$\partial_0(f_1 * f_2) = (\partial_0 f_1) * f_2 + f_1 * (\partial_0 f_2) ,$$

$$\partial_1(f_1 * f_2) = (\partial_1 f_1) * f_2 + (e^{-i\lambda\partial_0} f_1) * (\partial_1 f_2) ,$$

for all  $f_1, f_2 \in \text{Fun}^\infty(\mathbb{R}^2)$ .

### C. A Dirac operator for $\kappa$ -Minkowski space

The first attempt to define a Dirac operator would be to use the classical one  $\not{D}$ .

Properties (21) means that  $\not{D}$  has not bounded commutator with the algebra. Indeed:

$$[\partial_0, \Omega(f)] = \Omega(\partial_0 f) , \quad f \in \mathcal{A}^\infty ,$$

is bounded, but  $[\partial_1, \Omega(f)]$  is not, due to the presence of the unbounded operator  $e^{-i\lambda\partial_0}$ .

$D := i\partial_0$  has non-trivial sign (it is not positive), has dense domain in  $\mathcal{H}$  and bounded commutators with  $\mathcal{A}$ . Geometrically, the evaluation at  $\eta_1 = 0$ :

$$\Omega(f) \mapsto f(\eta_0, 0) ,$$

is an algebra morphism  $\mathcal{A} \rightarrow C_0(\mathbb{R})$ , and tells us that  $\mathbb{R}$  is a commutative subspace.  $D$  is just the Dirac operator on this subspace.

Let  $\Delta := -\partial_0^2 + 1$ . Intuitively, one would expect that the axiom (S1) in the definition of spectral triple is not satisfied. For  $\lambda = 0$ ,  $f \cdot \Delta^{-1/2}$  is not a compact operator on  $L^2(\mathbb{R}^2)$ . Surprisingly if  $\lambda \neq 0$ ,  $\Omega(f)\Delta^{-1/2}$  is a compact operator on  $\mathcal{H}_\mu$ , if  $\mu$  is a finite measure on  $\mathbb{R}$  (i.e.  $\int_{\mathbb{R}} d\mu(\eta_0) < \infty$ ) and absolutely continuous with respect to the Lebesgue measure.

**Proposition 7** *Let  $\mu$  be a finite measure on  $\mathbb{R}$ , absolutely continuous with respect to the Lebesgue measure, and let  $\lambda \neq 0$ . Then,  $(\mathcal{A}^\infty, \mathcal{H}_\mu, i\partial_0)$  is a  $1^+$ -summable spectral triple. The associated Dixmier trace is just the cyclic integral of [5]:*

$$\oint \Omega(f)\Delta^{-1/2} := \text{Res}_{z=1} \text{Trace}_{\mathcal{H}_\mu} \Omega(f)\Delta^{-z/2} \propto \int_{\mathbb{R} \times \mathbb{R}^*} f(\vec{x}) \frac{dx_0 dx_1}{|x_1|} , \quad (22)$$

where  $\Delta := -\partial_0^2 + 1$ .



*Proof:*

We decompose  $\mathcal{H}_\mu = \mathcal{V}_0 \otimes \mathcal{V}_1$ , with  $\mathcal{V}_0 := L^2(\mathbb{R}, d\mu(\eta_0))$  and  $\mathcal{V}_1 := L^2(\mathbb{R}^*, d\eta_1/|\eta_1|)$ .

Let us recall some fact from [20] (see also [21] for a pedagogical presentation and [10] for considerations on the non-compact case).

Let  $\mu'(\eta_0) = d\mu(\eta_0)/d\eta_0$ . By hypothesis  $\mu' \in C_0(\mathbb{R}^2)$ .

Being  $i\partial_0$  the Dirac operator on  $\mathbb{R}$ ,  $f\Delta^{-1/2}$  is compact on  $L^2(\mathbb{R}, d\eta_0)$  for all  $f \in C_0(\mathbb{R}^2)$ . In particular, taking  $f = \mu'$ , we prove that  $\Delta^{-1/2}$  is compact on  $\mathcal{V}_0$ .

From the formula:

$$\text{Res}_{z=1} \text{Trace}_{L^2(\mathbb{R}, d\eta_0)} f \Delta^{-z/2} \propto \int_{\mathbb{R}} f(\eta_0) d\eta_0 ,$$

(we don't care about the proportionality constant, which is independent on  $f$ ) we deduce

$$\text{Res}_{z=1} \text{Trace}_{\mathcal{V}_0} \Delta^{-z/2} = \text{Res}_{z=1} \text{Trace}_{L^2(\mathbb{R}, d\eta_0)} \mu' \Delta^{-z/2} \propto \int_{\mathbb{R}} d\mu(\eta_0) < \infty .$$

The operator  $\Delta^{-z/2}$  is traceclass on  $\mathcal{V}_0$  if  $z > 1$ , and in the Dixmier class  $\mathcal{L}^{1+}(\mathcal{V}_0)$  if  $z = 1$ . On  $\mathcal{H}_\mu$ , if  $f \in \mathcal{A}$ , the kernel of the operator  $\Omega(f)\Delta^{-z/2}$  is the distribution:

$$K_f(\vec{\eta}, \vec{\eta}') = \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2p \tilde{f}(\vec{p}) e^{i\vec{p}\vec{\eta}} \delta(\eta'_1 - \eta_1 e^{\lambda p_0}) \int_{\mathbb{R}} d\xi (1 + \xi^2)^{-z/2} e^{i\xi(\eta_0 - \eta'_0)} ,$$

where the integral in  $d\xi$  is the resolvent of  $\Delta^{z/2}$ .

We consider first the case in which  $f$  is integrable ( $f \in L^1(\mathbb{R}^*, d\eta_0 d\eta_1/|\eta_1|)$ ).

The partial trace of  $\Omega(f)\Delta^{-z/2}$  on  $\mathcal{V}_1$  is the operator with kernel:

$$\begin{aligned} \hat{K}_f(\vec{\eta}, \vec{\eta}') &= \int_{\mathbb{R}^*} \frac{d\eta_1}{|\eta_1|} K_f(\eta_0, \eta_1; \eta'_0, \eta_1) \\ &= \frac{\lambda^{-1}}{2\pi} \int_{\mathbb{R}^*} \frac{d\eta_1}{|\eta_1|} \int_{\mathbb{R}} dp_1 \tilde{f}(0, p_1) e^{ip_1 \eta_1} \int_{\mathbb{R}} d\xi (1 + \xi^2)^{-z/2} e^{i\xi(\eta_0 - \eta'_0)} \\ &= \frac{\lambda^{-1}}{2\pi} \left( \int_{\mathbb{R} \times \mathbb{R}^*} \frac{dx_0 dx_1}{|x_1|} f(\vec{x}) \right) \int_{\mathbb{R}} d\xi (1 + \xi^2)^{-z/2} e^{i\xi(\eta_0 - \eta'_0)} . \end{aligned}$$

Then, as operators on  $\mathcal{V}_0$ :

$$\text{Trace}_{\mathcal{V}_1} \{ \Omega(f) \Delta^{-z/2} \} = \frac{\lambda^{-1}}{2\pi} \left( \int_{\mathbb{R} \times \mathbb{R}^*} f(\vec{x}) \frac{dx_0 dx_1}{|x_1|} \right) \Delta^{-z/2} . \quad (23)$$

From what said above about  $\Delta^{-z/2}$ , we see that  $\Omega(f)\Delta^{-z/2}$  is traceclass on  $\mathcal{H}_\mu$  if  $z > 1$ , and in the Dixmier class  $\mathcal{L}^{1+}(\mathcal{H}_\mu)$  if  $z = 1$ .

Taking the trace on  $\mathcal{V}_0$  of (23) and then the residue in  $z = 1$  we prove (22).

Now, since  $\text{Fun}(\mathbb{R}^2) \subset L^2$ , integrable functions are dense in the algebra  $\mathcal{A}$ . So,  $\Omega(f)\Delta^{-1/2}$  is in the closure of  $\mathcal{L}^{1+}(\mathcal{H}_\mu)$  for all  $f \in \mathcal{A}$ . The closure of the Dixmier class are the compact operators  $\mathcal{K}$ , and this concludes the proof.  $\square$

**Remark:** The spectral triple constructed in this section has not a commutative analogue. Axiom (S1) is not satisfied for  $\lambda = 0$ .

### D. Equivariance properties of the representation

Let  $\mathcal{A}$  be the involutive algebra defined in proposition 6,  $\mathcal{H}$  the Hilbert space:

$$\mathcal{H} := \mathcal{H}_0 \otimes \mathbb{C}^2 = L^2(\mathbb{R} \times \mathbb{R}^*, |\eta_1|^{-1} d\eta_0 d\eta_1) \otimes \mathbb{C}^2 ,$$

and  $\gamma$  the grading in (11). We lift trivially the representation (16) of  $\mathcal{A}$  from  $\mathcal{H}_0$  to  $\mathcal{H}$ .

We postpone the problem of constructing an equivariant Dirac operator and study the equivariance properties of the data  $(\mathcal{A}, \mathcal{H}, \gamma)$ . Clearly  $\mathcal{A}$  commutes with  $\gamma$ , and so  $\gamma$  is a natural candidate for the grading.

The space  $\mathcal{H}_0$  carries a representation of the quantum Euclidean group, the Euclidean analogue of  $\kappa$ -Poincaré, which we indicate with  $U_\kappa(iso(2))$ . Recall that the Hopf algebra is generated by  $E, P, N$  with commutation relations:

$$[E, P] = 0 , \quad [N, P] = \frac{i}{2\lambda}(1 - e^{2\lambda E}) + \frac{i\lambda}{2}P^2 , \quad [N, E] = iP ,$$

coproduct:

$$\begin{aligned} \Delta E &= E \otimes 1 + 1 \otimes E , \\ \Delta P &= P \otimes 1 + e^{\lambda E} \otimes P , \\ \Delta N &= N \otimes 1 + e^{\lambda E} \otimes N , \end{aligned}$$

and counit/antipode:

$$\epsilon(E) = \epsilon(P) = \epsilon(N) = 0 , \quad S(E) = -E , \quad S(P) = -Pe^{-\lambda E} , \quad S(N) = -Ne^{-\lambda E} .$$

There is a natural representation of the  $(E, P)$  sub-Hopf-algebra of  $U_\kappa(iso(2))$  on a space dense in  $\mathcal{H}_0$ , defined by

$$\rho(E) = -i\partial_0 , \quad \rho(P) = -i\partial_1 .$$

This extend to a representation of the full Hopf-algebra if we define:

$$\begin{aligned} \rho(N) &= \eta_0 \rho(P) + \eta_1 \rho\left(\frac{1-e^{2\lambda E}}{2\lambda} + \frac{\lambda}{2}P^2\right) \\ &= -i\eta_0\partial_1 + \frac{1}{2\lambda}\eta_1(1 - e^{-2i\lambda\partial_0}) - \frac{\lambda}{2}\eta_1\partial_1^2 . \end{aligned} \quad (24)$$

On the space  $\mathbb{C}^2$ , we call  $\sigma$  the representation of the commutative  $\mathbb{R}$  subalgebra, defined by:

$$\sigma(N) := \frac{1}{2}\gamma = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma(E) = \sigma(P) = 0 .$$

On the space  $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$  we consider the representation  $\rho \otimes \sigma$  defined through the (opposite) coproduct:

$$(\rho \otimes \sigma)(h) = \rho(h_{(2)}) \otimes \sigma(h_{(1)}) , \quad \forall h \in U_\kappa(iso(2)) .$$

This representation commutes with the grading  $\gamma$ , since the image through  $\sigma$  of the algebra is in the subspace of  $\text{Mat}_2(\mathbb{C})$  spanned by 1 and  $\gamma$ .

Before discussing the  $U_\kappa(iso(2))$ -equivariance of  $(\mathcal{A}, \mathcal{H}, \gamma)$  we need to define a covariant action of the Hopf algebra on  $\mathcal{A}$ . If we define:

$$h \triangleright \Omega(f) := \Omega(\rho(h)f) , \quad \forall h \in U_\kappa(iso(2)), \Omega(f) \in \mathcal{A} ,$$

this is a representation of the Hopf-algebra. We want to prove that it is covariant, that is:

$$h \triangleright \Omega(f_1)\Omega(f_2) = \{h_{(1)} \triangleright \Omega(f_1)\} \{h_{(2)} \triangleright \Omega(f_2)\} ,$$

or equivalently using the  $*$ -product:

$$\rho(h)(f_1 * f_2) = \{\rho(h_{(1)})f_1\} * \{\rho(h_{(2)})f_2\} .$$

**Proposition 8** *The action  $\triangleright$  of  $U_\kappa(iso(2))$  on  $\mathcal{A}$  is covariant. Moreover:*

$$h \triangleright a = \rho(h_{(1)}) a \rho(Sh_{(2)}) ,$$

for all  $h \in U_\kappa(iso(2))$  and  $a \in \mathcal{A}$  (on a subspace dense in  $\mathcal{H}_0$ ).

*Proof:*

Using (20), we deduce that the covariance condition is equivalent to:

$$\rho(h)\Omega(f_1)f_2 = \Omega(\rho(h_{(1)})f_1) \cdot \{\rho(h_{(2)})f_2\} . \quad (25)$$

Since on both sides appear representations of  $U_\kappa(iso(2))$ , it is sufficient to do the check for the generators  $E, P, N$ .

We can rewrite (21) as:

$$\begin{aligned} \rho(E)\Omega(f_1)f_2 &= \Omega(\rho(E)f_1)f_2 + \Omega(f_1)\{\rho(E)f_2\} , \\ \rho(P)\Omega(f_1)f_2 &= \Omega(\rho(P)f_1)f_2 + \Omega(\rho(e^{\lambda E})f_1)\{\rho(P)f_2\} , \end{aligned}$$

and then the action of  $E$  and  $P$  is covariant.

In the same way, using (21) and (24), it is a straightforward computation to prove the covariance of the action of  $N$ .

If in equation (25) we replace  $h$  with  $h_{(1)}$ , call  $a = \Omega(f_1) \in \mathcal{A}$ ,  $f_2 = \rho(Sh_{(2)})\psi$ , and recall that  $\Omega(\rho(h_{(1)})f_1) = h_{(1)} \triangleright a$ , we obtain:

$$\rho(h_{(1)})a\rho(Sh_{(2)})\psi = (h_{(1)} \triangleright a)\rho(h_{(2)}S(h_{(3)}))\psi = \{\epsilon(h_{(2)})h_{(1)} \triangleright a\}\psi = (h \triangleright a)\psi .$$

This concludes the proof.  $\square$

Since the representation of  $\mathcal{A}$  is lifted diagonally from  $\mathcal{H}_0$  to  $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$ ,  $\sigma(h)a = a\sigma(h)$  for all  $h \in U_\kappa(iso(2))$  and  $a \in \mathcal{A}$ . Then, as a corollary:

$$h \triangleright a = (\rho \otimes \sigma)(h_{(1)}) a (\rho \otimes \sigma)(Sh_{(2)}) .$$

This means that:

**Corollary 9**  $(\mathcal{A}, \mathcal{H}, \gamma)$  is  $U_\kappa(iso(2))$ -equivariant.

## E. An $U_\kappa(iso(2))$ -equivariant Dirac operator

To have an equivariant spectral triple on  $\kappa$ -Minkowski space, it remain to find a Dirac operator  $D$  that is equivariant for the action of the quantum Euclidean group.

We write  $D$  as a formal pseudo-differential operator:

$$D\psi(\vec{\eta}) = \int_{\mathbb{R}^2} e^{i\vec{p}\vec{\eta}} \begin{pmatrix} 0 & T(\vec{\eta}, \vec{p}) \\ T(\vec{\eta}, \vec{p})^* & 0 \end{pmatrix} \tilde{\psi}(\vec{p}) d^2p , \quad \psi \in \mathcal{H} ,$$

and determine the symbol  $T$  imposing equivariance. The matrix form of the symbol is a consequence of the grading and formal selfadjointness.

Since  $(\rho \otimes \sigma)(E) = \rho(E)$ ,  $(\rho \otimes \sigma)(P) = \rho(P)$  and  $(\rho \otimes \sigma)(N) = \sigma(N) + \rho(N)\sigma(e^{\lambda E}) = \rho(N) + \frac{1}{2}\gamma$ , the equivariance conditions become:

$$[\rho(E), D] = 0 \quad , \quad [\rho(P), D] = 0 \quad , \quad [\rho(N), D] = -\frac{1}{2}[\gamma, D] \equiv -\gamma D \quad .$$

The first two conditions are equivalent to:

$$\partial_0 T(\vec{\eta}, \vec{p}) = \partial_1 T(\vec{\eta}, \vec{p}) = 0 \quad .$$

Then,  $T(\vec{\eta}, \vec{p}) = T(\vec{p})$ . The last condition can be written as:

$$[\chi(N), T(\vec{p})] = -T(\vec{p}) \quad , \quad [\chi(N), T(\vec{p})^*] = T(\vec{p})^* \quad , \quad (26)$$

where

$$\chi(N) = ip_1 \frac{\partial}{\partial p_0} + i \left( \frac{1 - e^{2\lambda p_0}}{2\lambda} + \frac{\lambda}{2} p_1^2 \right) \frac{\partial}{\partial p_1} \quad .$$

If we define  $\chi(E) = p_0$  and  $\chi(P) = p_1$ , then the map  $\chi$  extends to a representation of the  $U_\kappa(iso(2))$  Hopf algebra.

In [22] was constructed for the first time a (formal) isomorphism between  $\kappa$ -Poincaré and the Poincaré algebra. The Euclidean counterpart is the change of coordinates:

$$e^{-\lambda p_0} p_1 =: r \sin \theta \quad , \quad \frac{1}{\lambda} \sinh(\lambda p_0) + \frac{\lambda}{2} p_1^2 e^{-\lambda p_0} =: r \cos \theta \quad ,$$

with  $r \in \mathbb{R}_0^+$  and  $\theta \in S^1$ . The Casimir of the quantum Euclidean group is:

$$m^2 := \left( \frac{2}{\lambda} \sinh \frac{\lambda E}{2} \right)^2 + e^{-\lambda E} P^2 \quad .$$

It is related to  $r^2$  by:

$$r^2 = \chi \left( m^2 \left( 1 + \frac{\lambda^2 m^2}{4} \right) \right) \quad , \quad (27)$$

and then  $r^2$  is central and  $[\chi(N), r^2] = 0$ . So,  $\chi(N) = \tilde{N}(r, \theta) \partial_\theta$  is a derivation in the  $\theta$  direction, with  $\tilde{N}$  defined by  $\tilde{N}(r, \theta) r e^{i\theta} = -i[\chi(N), r e^{i\theta}]$ .

With a straightforward computation we arrive at  $[\chi(N), r e^{i\theta}] = r e^{i\theta}$ , and prove that:

$$\chi(N) = -i \frac{\partial}{\partial \theta} \quad .$$

The general solution of (26) is  $T(\vec{p}) = R(r) e^{-i\theta}$ , with  $R$  an arbitrary function. If we want the classical Dirac operator as  $\lambda = 0$  limit, we are forced to choose  $R(r) = -r$ , and the final solution is:

$$\begin{aligned} D &= -\rho \left( \begin{array}{cc} 0 & \frac{1}{\lambda} \sinh(\lambda E) + e^{-\lambda E} P \left( \frac{\lambda}{2} P - i \right) \\ \frac{1}{\lambda} \sinh(\lambda E) + e^{-\lambda E} P \left( \frac{\lambda}{2} P + i \right) & 0 \end{array} \right) \\ &= \left( \begin{array}{cc} 0 & \frac{1}{\lambda} \sinh(i\lambda \partial_0) + e^{i\lambda \partial_0} \left( 1 + \frac{\lambda}{2} \partial_1 \right) \partial_1 \\ \frac{1}{\lambda} \sinh(i\lambda \partial_0) - e^{i\lambda \partial_0} \left( 1 - \frac{\lambda}{2} \partial_1 \right) \partial_1 & 0 \end{array} \right) \quad . \end{aligned} \quad (28)$$

This Dirac operator is very similar to the one constructed in [3, 4], but for Euclidean signature instead of Lorentzian one, and 1 + 1 dimension instead of 3 + 1.

A deformed Leibniz rule for  $D$  comes from the coproduct of  $E, P$ , and tells us that commutators with the algebra are not bounded, due to the presence of the  $e^{\lambda E}$  factor. Using equation (2.5) of [4] one can reach the same conclusion for the Dirac operators in [3, 4].

From (27) we derive the following relation:

$$D^2 = \rho \left( m^2 \left( 1 + \frac{\lambda^2 m^2}{4} \right) \right) ,$$

that is the same as equation (12) in [3].

## VI. CONCLUSION

In quantum mechanics over  $\mathbb{R}^n$ , it is usual to work in momentum space by means of Fourier transform. If the physical system under consideration lives in a curved manifold, with trivial tangent bundle, “momenta” are globally defined as vector fields on the manifold, and in general do not commute. We have illustrated this situation with an example that is recurrent in physics, the deSitter space in three dimensions. In such a situation, if we want to work in “momentum space”, we need the tools of noncommutative geometry.

In these notes, we have studied the lowest dimensional non-trivial (i.e. non-commutative) example, when the manifold  $M$  is the connected component of the group of affine transformations of the real line, and found that the “dual” space is  $\kappa$ -Minkowski. This is the unique two-dimensional noncommutative example coming from a Lie group.

We have found a natural representation of  $\kappa$ -Minkowski on  $L^2(M, \mu)$ , with  $\mu$  the (left) Haar measure on the group  $M$ .

Usually this spacetime is studied from an Hopf-algebra point of view, considering  $U(\text{Lie } M)$  as the polynomial algebra of coordinates on some virtual space [2]. We have argued how, using Weyl quantization, it is possible to define the associated  $C^*$ -algebra. The definition of the  $C^*$ -algebra allows one to study the topology of the space, following the general philosophical viewpoint that  $C^*$ -algebra theory may be regarded as a kind of non-commutative topology.

In the last section, we have constructed a spectral triple associated to the cyclic integral of [5], and an  $U_\kappa(\text{iso}(2))$ -equivariant Dirac operator which does not satisfies the axioms for a spectral triple.

Apart for its intrinsic interest,  $\kappa$ -Minkowski is a simple non-trivial example in which to compare Connes approach to noncommutative geometry with the Hopf-algebraic one.

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## Appendix A: QUANTUM HOMOGENEOUS SPACES

Let  $G$  be a (compact) topological group that acts on a (compact, Hausdorff) topological space  $M$ ,  $m_0 \in M$  a fixed point and  $K \subset G$  the subgroup that leaves  $m_0$  invariant.  $M$  is called *homogeneous* if it consists of a single  $G$ -orbit (every two points can be connected by a transformation from  $G$ ). It is well known that in this case that the functions  $C(M)$  can be viewed as the subset of  $C(G)$  of  $K$ -invariant functions.

In the noncommutative case, when an Hopf algebra  $H$  coacts on an algebra  $A$ , we say that  $A$  is an *embeddable* space for  $H$  if there exists an inclusion  $j : A \hookrightarrow H$  that relates the coaction with the coproduct  $\Delta$  of  $H$  through one of the formulas:

$$\Delta \circ j = (id \otimes j) \circ \Delta_L \quad , \quad \Delta \circ j = (j \otimes id) \circ \Delta_R \quad ,$$

depending on which coaction one is considering: a left one  $\Delta_L$  or a right one  $\Delta_R$ .

Commutative homogeneous spaces  $C(M)$  are embeddable in  $C(G)$ . Then, embeddable noncommutative spaces provide a possible generalization for the notion of homogeneous space.

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